Growth of the magnetic field in Hall magnetohydrodynamics

2004 J. Phys. A: Math. Gen. 37 9317

(http://iopscience.iop.org/0305-4470/37/39/017)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 9317–9323 PII: S0305-4470(04)79338-5

Growth of the magnetic field in Hall magnetohydrodynamics

Manuel Núñez

Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain

E-mail: mnjmhd@am.uva.es

Received 16 April 2004, in final form 12 August 2004
Published 15 September 2004
Online at stacks.iop.org/JPhysA/37/9317
doi:10.1088/0305-4470/37/39/017

Abstract

While the Hall magnetohydrodynamics (MHD) model has been explored in depth in connection with the dispersive waves relevant in magnetic reconnection, a theoretical study of the mathematical features of this system is lacking. We consider here the boundedness of the solutions of the Hall MHD equations. With Dirichlet boundary conditions the total energy of the system is maintained, and dissipated by diffusion, but the behaviour of the higher moments of the magnetic field is more complicated. It is found that certain unusual geometries of the initial condition may lead to a blow-up of the L^3-norm of the field. Nevertheless, reasonable assumptions upon the correlation between the size of the magnetic field and the curvature of field lines imply that the magnetic field remains uniformly bounded.

PACS numbers: 52.30.Cv, 52.30.Ex, 96.60.Rd

1. Introduction

The Hall magnetohydrodynamics (MHD) system is in a certain sense a compromise between the two-fluid MHD equations, which should be necessary to describe phenomena where electrons and ions decouple, and the one-fluid classical MHD system, which is far simpler and more amenable to analysis. Its main field of application is magnetic reconnection, where the so-called whistlers, dispersive waves speeding electrons away from the reconnection sheet, play a prominent role in the modern description. We do not detail the modifications of Ohm's law yielding the Hall MHD equations: for this see, e.g., [1, 2]. For some numerical models of reconnection, see [3–6]; for the evolution of the Alfvén waves under the Hall MHD model, see [7]. We start with the Hall MHD normalized incompressible system

$$\frac{\partial \mathbf{v}}{\partial t} = \nu\Delta\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{J}\times\mathbf{B} - \nabla p \qquad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} - \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{v} - h \nabla \times (\mathbf{J} \times \mathbf{B}) \tag{2}$$

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0, \tag{3}$$

where $\mathbf{v}$ represents the (ion) velocity, $\mathbf{B}$ the magnetic field, $\mathbf{J} = \nabla \times \mathbf{B}$ the current density, p the kinetic pressure, ν the viscosity, η the resistivity and h (>0) the Hall constant. A bounded forcing could be added to the momentum equation (1) without modifying our conclusions. We will assume that all the magnitudes lie within a smooth bounded domain Ω, necessarily three-dimensional: the Hall term in equation (2) does not allow the magnetic field to remain within a plane, as happens in classical MHD.

Initial and boundary conditions must be added to the system. To study which ones are physically admissible, we demand that the total energy of the system must not grow in the absence of external inputs; in fact, it must decay due to the presence of dissipative terms. In the collisionless case ($\nu = \eta = 0$) the energy must remain constant.

2. Energy inequalities and boundary conditions

In order to avoid proliferation of boundary integrals, we impose a sticky boundary, i.e., $\mathbf{v} \mid_{\partial\Omega} = \mathbf{0}$; for this condition not to overdetermine the system, we need $\nu > 0$. Multiplying (1) by $\mathbf{v}$, (2) by $\mathbf{B}$ and integrating in Ω, we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \int_\Omega v^2 + B^2 \, \mathrm{d}V = D_B - \nu \int_\Omega |\nabla \mathbf{v}|^2 \, \mathrm{d}V - h \int_\Omega \nabla \times (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{B} \, \mathrm{d}V, \tag{4}$$

where

$$D_B = \eta \int_{\partial\Omega} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial n} \, \mathrm{d}\sigma - \eta \int_\Omega |\nabla \mathbf{B}|^2 \, \mathrm{d}V. \tag{5}$$

D_B may also be written in a different way: since $\Delta \mathbf{B} = -\nabla \times \mathbf{J}$,

$$D_B = -\eta \int_\Omega (\nabla \times \mathbf{J}) \cdot \mathbf{B} \, \mathrm{d}V = -\eta \int_{\partial\Omega} (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{n} \, \mathrm{d}\sigma - \eta \int_\Omega J^2 \, \mathrm{d}V, \tag{6}$$

although this does not mean that the volume and boundary integrals in (5) and (6) coincide, respectively, with each other.

Let us consider the Hall term

$$D_H = -h \int_\Omega \nabla \times (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{B} \, \mathrm{d}V. \tag{7}$$

This equals

$$D_H = -h \int_{\partial\Omega} ((\mathbf{J} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{n} \, \mathrm{d}\sigma - h \int_\Omega (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{J} \, \mathrm{d}V, \tag{8}$$

and the volume integral vanishes.

The condition to avoid inputs of energy from the outside with the second form (6) of D_B and D_H would be

$$(\mathbf{J} \times \mathbf{B}) \cdot \mathbf{n} \mid_{\partial\Omega} = 0 \tag{9}$$

$$((\mathbf{J} \times \mathbf{B}) \times \mathbf{B}) \cdot \mathbf{n} \mid_{\partial\Omega} = 0. \tag{10}$$

This may be achieved by setting either $\mathbf{B} = \mathbf{0}$, $\mathbf{J} = \mathbf{0}$ or $\mathbf{J} \| \mathbf{B}$ in the boundary $\partial\Omega$. However, this implies only

$$\frac{1}{2} \frac{\partial}{\partial t} \int_\Omega v^2 + B^2 \, \mathrm{d}V + \int_\Omega \nu |\nabla \mathbf{v}|^2 + \eta J^2 \, \mathrm{d}V = 0, \tag{11}$$

and from this it follows that $v \to 0$, but not that $B \to 0$; the norms $\|J\|_2$ and $\|\nabla\mathbf{B}\|_2$ are only equivalent if $\mathbf{B}\cdot\mathbf{n} = 0$ at $\partial\Omega$. $\mathbf{B}$ could become a nonzero gradient. In the collisionless case, where we do not demand the dissipation of the field, the above boundary conditions are acceptable.

This problem does not arise with the first form of D_B: we must have

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial n}\bigg|_{\partial\Omega} = 0, \tag{12}$$

in addition to (10). Obviously, the simplest way to achieve this is to set the homogeneous condition

$$\mathbf{B}|_{\partial\Omega} = \mathbf{0}, \tag{13}$$

which we assume from now on.

3. Moments of the magnetic field

The energy is not the only measure of the field to be considered. Higher-order moments of the form $\int_\Omega B^k\,\mathrm{d}V$, $k > 2$, provide finer estimates of the size of $\mathbf{B}$, culminating in the maximum norm $\|\mathbf{B}\|_\infty$. The function B^k is certainly differentiable, even at the points where $\mathbf{B}$ vanishes, if $k \geqslant 2$. We therefore consider, for $p \geqslant 1$, the evolution of

$$\|\mathbf{B}\|_{2p}^{2p} = \int_\Omega B^{2p}\,\mathrm{d}V. \tag{14}$$

Since

$$\frac{\partial}{\partial t}B^{2p} = 2pB^{2p-2}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}, \tag{15}$$

by using the induction equation we obtain

$$\begin{aligned}\frac{1}{2p}\int_\Omega B^{2p}\,\mathrm{d}V &= -\frac{1}{2p}\int_\Omega \mathbf{v}\cdot\nabla B^{2p}\,\mathrm{d}V + \eta\int_\Omega B^{2p-2}\mathbf{B}\cdot\Delta\mathbf{B}\,\mathrm{d}V\\ &\quad + \int_\Omega B^{2p-2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{v})\,\mathrm{d}V - h\int_\Omega B^{2p-2}\mathbf{B}\cdot(\nabla\times(\mathbf{J}\times\mathbf{B}))\,\mathrm{d}V.\end{aligned} \tag{16}$$

The first term of the right-hand side integrates to zero. Let us denote the remaining three terms, respectively, by A_1, A_2 and A_3. By writing A_1 in a form appropriate to the use of Gauss's theorem, we find

$$\begin{aligned}A_1 &= \eta\int_\Omega B^{2p-2}\mathbf{B}\cdot\Delta\mathbf{B}\,\mathrm{d}V\\ &= \eta\int_\Omega\sum_j(\nabla\cdot(B_jB^{2p-2}\nabla B_j) - \nabla B_j\cdot(B^{2p-2}B_j))\,\mathrm{d}V\\ &= \frac{\eta}{2}\int_{\partial\Omega}B^{2p-2}\frac{\partial B^2}{\partial n}\,\mathrm{d}\sigma - \eta\int_\Omega B^{2p-2}|\nabla\mathbf{B}|^2\,\mathrm{d}V - \frac{\eta(p-1)}{2}\int_\Omega B^{2p-4}|\nabla B^2|^2\,\mathrm{d}V,\end{aligned} \tag{17}$$

and the boundary integral is zero. As for A_2, by a similar argument

$$\begin{aligned}A_2 &= \int_\Omega B^{2p-2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{v})\,\mathrm{d}V\\ &= \int_\Omega \mathbf{B}\cdot\nabla(B^{2p-2}\mathbf{v}\cdot\mathbf{B}) - \mathbf{v}\cdot(\mathbf{B}\cdot\nabla(B^{2p-2}\mathbf{B}))\,\mathrm{d}V\\ &= \int_{\partial\Omega}B^{2p-2}(\mathbf{v}\cdot\mathbf{B})\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}\sigma - \int_\Omega B^{2p-2}\mathbf{v}\cdot(\mathbf{B}\cdot\nabla\mathbf{B}) + (p-1)B^{2p-4}(\mathbf{v}\cdot\mathbf{B})(\mathbf{B}\cdot\nabla B^2)\,\mathrm{d}V,\end{aligned} \tag{18}$$

and again the boundary integral vanishes. This term may be bounded in the following form: assuming that the velocity $\mathbf{v}$ is uniformly bounded,

$$|A_2| \leqslant \|\mathbf{v}\|_\infty \left(\int_\Omega B^{2p-1} |\nabla \mathbf{B}| + (p-1) B^{2p-2} |\nabla B^2| \, \mathrm{d}V \right),$$

so that by the inequality of Cauchy–Schwarz,

$$|A_2| \leqslant \|\mathbf{v}\|_\infty \left(\int_\Omega B^{2p} \, \mathrm{d}V \right)^{1/2} \left(\int_\Omega B^{2p-2} |\nabla \mathbf{B}|^2 \, \mathrm{d}V \right)^{1/2} + (p-1) \|\mathbf{v}\|_\infty \left(\int_\Omega B^{2p} \, \mathrm{d}V \right)^{1/2} \left(\int_\Omega B^{2p-4} |\nabla B^2|^2 \, \mathrm{d}V \right)^{1/2},$$

and by Young's inequality (or simply by $ab \leqslant a^2/2 + b^2/2$)

$$|A_2| \leqslant \frac{1}{4\eta} \|\mathbf{v}\|_\infty^2 \int_\Omega B^{2p} \, \mathrm{d}V + \eta \int_\Omega B^{2p-2} |\nabla \mathbf{B}|^2 \, \mathrm{d}V + \frac{p-1}{\eta} \|\mathbf{v}\|_\infty^2 \int_\Omega B^{2p} \, \mathrm{d}V + \frac{\eta(p-1)}{4} \int_\Omega B^{2p-4} |\nabla B^2|^2 \, \mathrm{d}V. \tag{19}$$

Therefore,

$$|A_1 + A_2| \leqslant \left(\frac{1}{4\eta} + \frac{p-1}{\eta} \right) \|\mathbf{v}\|_\infty^2 \int_\Omega B^{2p} \, \mathrm{d}V - \frac{\eta(p-1)}{4} \int_\Omega B^{2p-4} |\nabla B^2|^2 \, \mathrm{d}V. \tag{20}$$

Let us study now A_3. By elementary vectorial identities,

$$\begin{aligned}
A_3 &= -h \int_\Omega B^{2p-2} \mathbf{B} \cdot (\nabla \times (\mathbf{J} \times \mathbf{B})) \, \mathrm{d}V \\
&= -h \int_\Omega (\mathbf{J} \times \mathbf{B}) \cdot \nabla \times (B^{2p-2} \mathbf{B}) \, \mathrm{d}V \\
&= -h \int_\Omega (\mathbf{J} \times \mathbf{B}) \cdot (\nabla B^{2p-2} \times \mathbf{B} + B^{2p-2} \mathbf{J}) \, \mathrm{d}V \\
&= -h \int_\Omega (\mathbf{J} \times \mathbf{B}) \cdot (\nabla B^{2p-2} \times \mathbf{B}) \, \mathrm{d}V \\
&= -h \int_\Omega \left(\mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla B^2 \right) \cdot ((p-1) B^{2p-4} \nabla B^2 \times \mathbf{B}) \, \mathrm{d}V \\
&= -h(p-1) \int_\Omega B^{2p-4} (\mathbf{B} \cdot \nabla \mathbf{B}) \cdot (\nabla B^2 \times \mathbf{B}) \, \mathrm{d}V \\
&= h(p-1) \int_\Omega B^{2p-4} \nabla B^2 \cdot ((\mathbf{B} \cdot \nabla \mathbf{B}) \times \mathbf{B}) \, \mathrm{d}V.
\end{aligned} \tag{21}$$

Let us write this in terms of the parameters of the geometry of the magnetic field line. At the points where $\mathbf{B} = \mathbf{0}$, the integrand vanishes; in the remaining points, denote by $\mathbf{T}$ the unitary field vector, by $\mathbf{N}$ the normal and by $\mathbf{W}$ the binormal one; let s be the arc length parameter and κ the curvature. Then

$$\mathbf{B} \cdot \nabla \mathbf{B} = B \frac{\mathrm{d}}{\mathrm{d}s} (B\mathbf{T}) = \frac{1}{2} \frac{\mathrm{d}B^2}{\mathrm{d}s} \mathbf{T} + B^2 \kappa \mathbf{N}. \tag{22}$$

Thus,

$$(\mathbf{B} \cdot \nabla \mathbf{B}) \times \mathbf{B} = B^3 \kappa \mathbf{N} \times \mathbf{T} + \frac{1}{2} \frac{\mathrm{d}B^2}{\mathrm{d}s} \mathbf{T} \times \mathbf{T} = -B^3 \kappa \mathbf{W}. \tag{23}$$

Hence, we may write A_3 in the following forms:

$$A_3 = -h(p-1) \int_\Omega B^{2p-1} \kappa \mathbf{W} \cdot \nabla B^2 \, \mathrm{d}V \tag{24}$$

$$A_3 = -\frac{h(p-1)}{p+1/2}\int_\Omega B^{2p+1}\nabla\cdot(\kappa\mathbf{W})\,\mathrm{d}V. \tag{25}$$

Let us insist that at the points where the field vanishes and there is no binormal vector the integrand is also zero.

4. Possibility of blow-up

Let us assume that the velocity remains bounded uniformly in time. Although this does not follow from the original equations, it is physically very reasonable. It may be asked why this is a more plausible assumption than considering directly the magnetic field as bounded. Apart from general considerations on the motion of material points, the fact is that MHD (and Hall MHD) use the Faraday approximation to the Maxwell equations (taking the displacement current as zero), which does not hold for very fast motion of electric charges. Thus, if we admit Hall MHD as a valid hypothesis, it would be inconsistent not to take a bounded plasma velocity.

Let us also assume that the geometry of the field lines is such that $\nabla\cdot(\kappa\mathbf{W}) < \alpha < 0$ for an interval of time whose length we will specify later. Take for instance $p = 3/2$. Since by (20)

$$|A_1 + A_2| \leqslant \frac{3}{4}\frac{\|\mathbf{v}\|_\infty^2}{\eta}\int_\Omega B^3\,\mathrm{d}V, \tag{26}$$

and by (25) and our hypothesis

$$A_3 \geqslant \frac{h\alpha}{4}\int_\Omega B^4\,\mathrm{d}V, \tag{27}$$

equation (16) implies

$$\frac{\partial}{\partial t}\int_\Omega B^3\,\mathrm{d}V \geqslant -\frac{9\|\mathbf{v}\|_\infty^2}{4\eta}\int_\Omega B^3\,\mathrm{d}V + \frac{3h\alpha}{4}\int_\Omega B^4\,\mathrm{d}V. \tag{28}$$

Let us denote by $m(\Omega)$ the volume of Ω. By Hölder's inequality,

$$\int_\Omega B^3\,\mathrm{d}V \leqslant m(\Omega)^{1/4}\left(\int_\Omega B^4\,\mathrm{d}V\right)^{3/4}. \tag{29}$$

Hence,

$$\frac{\partial}{\partial t}\int_\Omega B^3\,\mathrm{d}V \geqslant -\frac{9\|\mathbf{v}\|_\infty^2}{4\eta}\int_\Omega B^3\,\mathrm{d}V + \frac{3h\alpha}{4}m(\Omega)^{-1/3}\left(\int_\Omega B^3\,\mathrm{d}V\right)^{4/3}. \tag{30}$$

Let us denote the integral of B^3 by F. Equation (30) is a differential inequality of the form

$$F' \geqslant -kF + cF^{4/3}. \tag{31}$$

Any solution of this inequality with initial condition $F(0) > (k/c)^3$ tends to ∞ at a finite time prior to

$$t_\infty = \int_{F(0)}^{\infty}\frac{\mathrm{d}x}{cx^{4/3} - kx}. \tag{32}$$

Therefore, the L^3-norm of $\mathbf{B}$ tends to infinity at a finite time. Obviously, the catch in this proof is the assumption $\nabla\cdot(\kappa\mathbf{W}) < \alpha$ for $t \in [0, t_\infty)$: we may easily imagine an initial condition where this condition occurs, but in principle we cannot assume that it will continue to hold for the necessary time. Note, however, that it depends only on the geometry of the field lines, not on the size of the magnetic field, whereas the time for blow-up decreases as $F(0)$ increases, and F is a function of the size B alone. Thus, we may imagine such a large $F(0)$ that the blow-up occurs before the evolution of the field has the time to modify enough the divergence of $\kappa\mathbf{W}$. While this argument is not completely tight, it shows that the possibility of blow-up cannot be excluded for a certain moment of the magnetic field in the Hall MHD.

5. Uniform boundedness of the field

The condition $\nabla\cdot(\kappa\mathbf{W}) < \alpha < 0$ is, however, highly artificial and none of the usual configurations of field lines satisfies it. Instead we find that there is an inverse correlation of the size of the magnetic field with the curvature of the field lines. There are several reasons for this: the easiest to understand is that dissipation tends to smooth things and cannot allow rapid changes of direction of a large field. Explicitly, since $\|\nabla\mathbf{B}\|_2^2$ is integrable in time, it must be small for large times. Since

$$|\nabla\mathbf{B}|^2 \geqslant \left|\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}s}\right|^2 \geqslant (B^2\kappa)^2, \tag{33}$$

$B^2\kappa$ must be small in the mean.

There also exists a more important, purely kinetic, reason for the anticorrelation of B and κ. Large-scale advecting flows tend to align the field in sheets where it is roughly parallel, and it points in opposite directions in adjacent sheets. This folding property has been studied in the context of kinematic dynamo theory with prescribed flows (see, e.g., [8, 9]), as well as with random δ-correlated flows [10]. The field is much smaller at the points where the field lines are strongly curved to cross from one sheet to another; independently of the diffusion, it is found that $B\kappa \sim 1$. It is also intuitive that a large magnetic field imposes a certain rigidity to the field line and makes it harder to curve. It must be noted that, while these magnetic geometries have been found in the context of classical MHD, they are stable in the Hall MHD and therefore likely to occur whenever the flow is chaotic. As a simple model, consider a field of the form $\mathbf{B} = (f(z), 0, 0)$, representing plane sheets where the field direction varies vertically. Then,

$$\mathbf{J}\times\mathbf{B} = -\big(0, 0, \tfrac{1}{2}(f(z)^2)\big), \tag{34}$$

so that the Hall term $\nabla\times(\mathbf{J}\times\mathbf{B})$ vanishes.

This prompts us to postulate that there exists a constant M such that for all time, $\|\kappa B\|_\infty \leqslant M$. We will see that with this condition, the magnetic field remains uniformly bounded for all time. We have, by (24)

$$\begin{aligned}
|A_3| &\leqslant h(p-1)\int_\Omega B^{2p-2}B\kappa|\nabla B^2|\,\mathrm{d}V \\
&\leqslant h(p-1)M\int_\Omega B^{2p-2}|\nabla B^2|\,\mathrm{d}V = h(p-1)M\int_\Omega B^pB^{p-2}|\nabla B^2|\,\mathrm{d}V \\
&\leqslant h(p-1)M\left(\int_\Omega B^{2p}\,\mathrm{d}V\right)^{1/2}\left(\int_\Omega B^{2p-4}|\nabla B^2|^2\,\mathrm{d}V\right)^{1/2} \\
&\leqslant \frac{2}{\eta}h^2(p-1)M^2\int_\Omega B^{2p}\,\mathrm{d}V + \frac{\eta}{8}(p-1)\int_\Omega B^{2p-4}|\nabla B^2|\,\mathrm{d}V.
\end{aligned} \tag{35}$$

Compensating the second term with the one in the bound on $|A_1 + A_2|$ (20), we get

$$\begin{aligned}
\frac{1}{2p}\frac{\partial}{\partial t}\int_\Omega B^{2p}\,\mathrm{d}V &\leqslant -\frac{\eta}{8}(p-1)\int_\Omega B^{2p-4}|\nabla B^2|^2\,\mathrm{d}V \\
&\quad + \frac{1}{\eta}\left(\|\mathbf{v}\|_\infty^2\left(p+\frac{1}{2}\right) + 2h^2(p-1)M^2\right)\int_\Omega B^{2p}\,\mathrm{d}V.
\end{aligned} \tag{36}$$

Hence, denoting $G_{2p} = \|\mathbf{B}\|_{2p}^{2p}$, we have

$$\frac{\partial}{\partial t} G_{2p} \leqslant -\frac{\eta}{4} p(p-1) \int_\Omega B^{2p-4} |\nabla B^2|^2 \,\mathrm{d}V + \frac{1}{\eta} \left(\|\mathbf{v}\|_\infty^2 p(2p+1) + 4h^2 p(p-1) M^2 \right) G_{2p}. \tag{37}$$

A somewhat involved argument using the Sobolev–Gagliardo inequalities, and which may be found in a different context in [11], proves that this inequality implies

$$\sup_{0\leqslant t\leqslant T} \|\mathbf{B}(t)\|_\infty \leqslant C\eta^{-3/2} \sup_{0\leqslant t\leqslant T} (\|\mathbf{v}(t)\|_\infty + Mh)^{3/2} \left(\|\mathbf{B}(0)\|_\infty + \sup_{0\leqslant t\leqslant T} \|\mathbf{B}(t)\|_2 \right), \tag{38}$$

for some constant C depending only on Ω. Since we already know that the magnetic energy remains bounded, the same may be said of the supremum norm of $\mathbf{B}$ and therefore of all its moments, provided, obviously, that the initial condition is bounded.

6. Conclusions

The equations of the Hall magnetohydrodynamics have not yet been studied from the viewpoint of the mathematical properties of its solutions. Homogeneous Dirichlet boundary conditions for the velocity and the magnetic field guarantee that the total energy is maintained by the advective terms and dissipated by ohmic and viscous diffusion. However, the behaviour of the higher moments of the field is more doubtful. In fact, a rather artificial geometry of magnetic field lines as initial condition may cause a blow-up of the L^3-norm of the magnetic field. Nevertheless, some physically plausible assumptions yield a magnetic field uniformly bounded in time. These are that the plasma velocity remains bounded and that the product of the size of the field times the curvature of the field line is also bounded. The last condition has been proved to hold in a large number of models.

References

[1] Vasyliunas V M 1975 Theoretical models of magnetic field line merging *Rev. Geophys. Space Phys.* **13** 303–36

[2] Craig I J D and Watson P G 2003 Magnetic reconnection solutions based on a generalized Ohm's law *Solar Phys.* **214** 131–50

[3] Drake J F, Biskamp D and Zeiler A 1997 Breakup of the electron current layer during 3D collisionless magnetic reconnection *Geophys. Res. Lett.* **24** 2921–4

[4] Yamada M, Ji H, Hsu S, Carter T A, Kulsrud R M and Trintchouk F 2000 Experimental investigation of the neutral sheet profile during magnetic reconnection *Phys. Plasmas* **7** 1781–7

[5] Rogers B N, Denton R E, Drake J F and Shay M A 2001 Role of dissipative waves in collisionless magnetic reconnection *Phys. Rev. Lett.* **87** 195004 1–4

[6] Shay M A, Drake J F, Rogers B N and Denton R E 2001 Alfvénic collisionless magnetic reconnection and the Hall term *J. Geophys. Res.* **106** 3759–72

[7] Laveder D, Passot T and Sulem P L 2002 Transverse dynamics of dispersive Alfvén waves: I. Direct numerical evidence of filamentation *Phys. Plasmas* **9** 293–304

[8] Du Y and Ott E 1993 Fractal dimensions of fast dynamo fields *Physica* D **67** 387–417

[9] Du Y and Ott E 1993 Growth rates for fast kinematic dynamo instabilities of chaotic fluid flows *J. Fluid Mech.* **257** 265–88

[10] Schekochihin S, Cowley S, Maron J and Malyshkin L 2001 Structure of small-scale magnetic fields in the kinematic dynamo theory *Phys. Rev.* E **65** 016305 1–17

[11] Núñez M 2003 Uniform boundedness of the magnetic field in a resistive plasma *Math. Meth. Appl. Sci.* **26** 171–7